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Generalized Forms, Connections, and Gauge Theories

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Generalized differential forms of type $N = 2$, and flat generalized connections are used to describe the $SO(p, q)$ form of Cartan's structure equations for metric geometries, source-free Yang-Mills fields, and the Einstein–Yang-Mills equations in four dimensions. Maxwell's equations for type $N = 2$ forms are also constructed.

KEY WORDS: generalized differential forms; gauge theory.

1. INTRODUCTION

In recent years various generalizations of the standard exterior algebra and calculus of differential forms have been discussed in the literature. An illustrative sample of such studies (Asada, 2001; Dubois-Violette, 1999, 2000; Cotrill-Shepherd and Naber, 2001, 2003; Madore, 1999) also contains references to related research. This paper is concerned with the further consideration of another generalization, one in which the algebra and calculus of ordinary exterior forms have been extended to an algebra and calculus of different *types of generalized differential forms*. The approach followed here has been developed in a number of papers (Nurowski and Robinson, 2001, 2002; Robinson, 2003), and applied to a variety of physical systems including field theories (Guo *et al.*, 2002).

Ordinary differential forms are, by definition, generalized forms of type $N = 0$. The expression for a single generalized form of type N, where N is a nonnegative integer, will usually include ordinary forms of different degrees. For example, a generalized p-form of type N may include forms of degree q, where $p \leq q \leq N + p$. Generalized forms of type N admit a number of different representations (Robinson, 2003). In this paper they will be described by using expansions which include N linearly independent minus 1-forms, ζ^i, $i = 1, \ldots, N$, and their exterior products. These minus 1-forms are required to satisfy

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the usual rules obeyed by ordinary p-forms, but for them $p = -1$. In order to ensure that their exterior derivatives are zero-forms and that $d^2 = 0$, they are required to satisfy the condition $d\zeta^i = k^i$, where k^i are constants. Although there is a freedom in the choice of the basis of minus 1-forms, by a GL(N) transformation, and this may be used to change the constants k^i and certain components, it will be assumed here that the basis is fixed and all the constants are nonzero. Consequently a generalized p-form of type N will be written in this paper as $\overset{p}{\mathbf{a}} = \overset{p}{\alpha} + \overset{p+1}{\alpha}{}_{i_1}\zeta^{i_1} + \overset{p+2}{\alpha}{}_{i_1 i_2}\zeta^{i_1}\zeta^{i_2} + \cdots + \overset{p+j}{\alpha}{}_{i_1,\ldots,i_j}\zeta^{i_1},\ldots,\zeta^{i_j} + \cdots + \overset{p+N}{\alpha}{}_{i_1,\ldots,i_N}\zeta^{i_1},\ldots,\zeta^{i_N}$. Here $\overset{p}{\alpha}$, $\overset{p+1}{\alpha}{}_{i_1,\ldots,}$ $\overset{p+j}{\alpha}{}_{i_1,\ldots,i_j} = \overset{p+j}{\alpha}{}_{[i_1,\ldots,i_j],\ldots,}$ $\overset{p+N}{\alpha}{}_{i_1,\ldots,i_N}$, are respectively, ordinary p-, $p+1$-,..., $p+j$-, ..., $p+N$- forms, $\mathbf{1} \leq j \leq N$, and $i_1, \ldots, i_j, \ldots, i_N$ range and sum over 1 to N. Each separate term in such an expansion is a generalized form of degree p. As in previous papers, bold-face Roman letters are used to denote generalized forms, ordinary forms are always denoted by Greek letters, and, where it is useful, the degree of a form is indicated above it. In this paper the exterior product of any two forms, for example $\alpha \wedge \beta$, is written $\alpha\beta$. By standard convention any ordinary form $\overset{q}{\alpha}$, with q either negative or greater than n, the dimension of the manifold M, is zero. When $p \geq 0$ a generalized form, such as $\overset{p}{\mathbf{a}}$ above, may be regarded as an extension of the ordinary p-form given by $\overset{p}{\alpha}$. Generalized forms of all different degrees and types obey the same basic rules of exterior multiplication and differentiation as those which govern the algebra and calculus of ordinary differential forms. Two basic results are, $\overset{p}{\mathbf{a}}\overset{q}{\mathbf{b}} = (-1)^{pq}\overset{q}{\mathbf{b}}\overset{p}{\mathbf{a}}$ and $d(\overset{p}{\mathbf{a}}\,\overset{q}{\mathbf{b}}) = d(\overset{p}{\mathbf{a}})\overset{q}{\mathbf{b}} + (-\mathbf{1})^p\overset{p}{\mathbf{a}}\,d(\overset{q}{\mathbf{b}})$. There are some differences from the standard results for ordinary, that is $N = 0$, forms when $N > 0$. For instance, it follows from the definitions that generalized forms of negative degree, $p \leq 0$, are permitted, and it has been shown that when $N > 0$ generalized forms are closed if and only if they are exact.

In this paper the focus will be on further applications of generalized connections of type $N = 2$ (Robinson, 2003). The primary aim is to show that Cartan's, Einstein's, and the Yang-Mills equations can all be formulated using flat generalized connections. First, results on type $N = 2$ generalized connections will be reviewed and the notation to be used in this paper will be fixed. Then it will be shown that Cartan's structure equations for metric connections, on an n-dimensional manifold M, can be simply represented in terms of a type $N = 2$ flat generalized connection. In contrast to an earlier approach using type $N = 1$ forms (Nurowski and Robinson, 2001) a formulation using a basis of 2-forms, rather than a co-frame, is employed here. Special cases, such as Ricci flat Levi-Civita connections, will also be noted and the results hold in any dimension. Next, a result which applies in four-dimensional space–times, and which is suggested by the formulation of Cartan's equations, will be exhibited. It will be shown how both the source free Yang-Mills equations and gravity coupled to the source free Yang-Mills field, via the Einstein–Yang-Mills equations, can be formulated in terms of type $N = 2$ flat generalized connections. These results provide a "universal" conceptual framework, that of flat connections constructed from generalized forms, within which all these equations can be placed. Finally, in order to provide a simple contrasting example which employs a nonflat connection, source-free Maxwell-like equations are constructed using type $N = 2$ forms on a four-dimensional manifold with a Lorentzian metric. The resulting equations can be reduced, by choice of gauge, to a Proca-like equation for an ordinary 3-form.

2. GENERALIZED CONNECTION FORMS OF TYPE $N = 2$

It will be helpful to think of generalized connections as extensions of ordinary connections, as follows. Let $\overset{1}{\alpha}$ be an ordinary ($N = 0$) connection 1-form with values in the Lie algebra $\mathfrak{g}$ of a Lie group G. Let $\overset{2}{\Omega}$ denote its curvature 2-form, so that $\overset{2}{\Omega} = d\overset{1}{\alpha} + \overset{1}{\alpha}\overset{1}{\alpha}$ and let d_1 denote the covariant exterior derivative with respect to $\overset{1}{\alpha}$. Define $\mathbf{A}$, a $\mathfrak{g}$-valued generalized connection 1-form of type N, to be the generalized extension of $\overset{1}{\alpha}$ when

$$\mathbf{A} = \overset{1}{\alpha} + \overset{2}{\alpha}_{i_1}\zeta^{i_1} + \overset{3}{\alpha}_{i_1}i_2\zeta^{i_1}\zeta^{i_2} + \overset{4}{\alpha}_{i_1}i_2i_3\zeta^{i_1}\zeta^{i_2}\zeta^{i_3} + \ldots \overset{N+1}{\alpha}{}_{i_1,\ldots,iN}\,\zeta^{i_1},\ldots,\zeta^{i_N}.$$

The ordinary p-forms, $2 \leq p \leq N + 1$, given by $\overset{2}{\alpha}_{i_1}, \ldots, \overset{N+1}{\alpha}{}_{i_1}, \ldots, i_N$ take values in $\mathfrak{g}$. The curvature 2-form of $\mathbf{A}$ is $\mathbf{F} = d\mathbf{A} + \mathbf{AA}$, and straightforward scomputation gives

$$\mathbf{F} = \overset{2}{\Omega} + \overset{2}{\alpha}_{i_1}k^{i_1} + \big(d_1\overset{2}{\alpha}_{i_1} + 2\overset{3}{\alpha}_{i_1i_2}k^{i_2}\big)\zeta^{i_1}$$
$$+ \big(d_1\overset{3}{\alpha}_{i_1i_2} + \overset{2}{\alpha}_{i_1}\overset{2}{\alpha}_{i_2} + 3\overset{4}{\alpha}_{i_1i_2i_3}k^{i_3}\big)\zeta^{i_1i_2} + \ldots$$

Only type $N = 2$ forms will be considered henceforth in this paper so the notation can be simplified a little. Let

$$\mathbf{A} = \alpha + \beta_i\zeta^i + \gamma\zeta^{12}, \tag{1}$$

$i = 1, 2$, be a generalized connection 1-form on a manifold M with values in the Lie algebra $\mathfrak{g}$ of a matrix Lie group G. Here α, β_i, and γ are $\mathfrak{g}$-valued ordinary 1-, 2-, and 3-forms respectively and $\zeta^{12} = \zeta^1\zeta^2$. The curvature 2-form is given by

$$\mathbf{F} = (\Omega_\alpha + \beta_i k^i) + (d_\alpha\beta_1 + k^2\gamma)\zeta^1 + (d_\alpha\beta_2 - k^1\gamma)\zeta^2$$
$$+ (d_\alpha\gamma + \beta_1\beta_2 - \beta_2\beta_1)\zeta^{12}, \tag{2}$$

and d_α denotes the covariant exterior derivative with respect to α, with the latter interpreted as an ordinary connection 1-form with curvature 2-form $\Omega_\alpha \equiv d\alpha + \alpha\alpha$. Such a convention will be followed throughout. By considering the case when the curvature $\mathbf{F}$ is zero it is easy to see from Eq. (2) that a flat connection can always be written just in terms of a 1-form and a 2-form, for example as

$$\mathbf{A} = \alpha - (k^1)^{-1}(\Omega_\alpha + \beta_2 k^2)\zeta^1 + \beta_2\zeta^2 + (k^1)^{-1}d_\alpha\beta_2\zeta^{12}. \tag{3}$$

As in previous papers, gauge transformations for such connections and curvatures may be defined to be generated by generalized zero-forms on M taking values in a Lie group **G**. This is a broad notion of gauge equivalence. The group **G** is determined by the semidirect product of a Lie group G and the abelian groups, under addition, of modules of certain ordinary differential forms with values in the Lie algebra $\mathfrak{g}$ of G. In the present context type $N = 2$ forms and connections are being considered so, without loss of essential generality, here $\mathbf{g} \in \mathbf{G}$ will be assumed to be of the form $\mathbf{g} = \mathbf{mn}$, where $\mathbf{m} = 1 + \mu\zeta^1 + \nu\zeta^{12}$ and $\mathbf{n} = \pi$. Here μ and ν are, respectively, ordinary 1- and 2-forms on M with values in the Lie algebra $\mathfrak{g}$ and π is an ordinary zero-form on M with values in the Lie group G.

[In previous papers, $\mathbf{g} \in \mathbf{G}$ was written in the form $\mathbf{g} = \pi(1 + \overset{1}{\pi}\zeta^1 \ldots) \in \mathbf{G}$, where π is an ordinary zero-form on M with values in the Lie group G, and $\overset{1}{\pi}$ is a 1-form with values in the Lie algebra $\mathfrak{g}$, etc. However the expression for **g** can always be rewritten differently as the product of two zero-forms on M, as $\mathbf{g} = \mathbf{mn}$, where $\mathbf{m} = 1 + \pi\overset{1}{\pi}\pi^{-1}\zeta^1 + \cdots$, and $\mathbf{n} = \pi$. Because this way of writing **g** enables the transformation properties of the connection to be expressed in a more readily recognizable form than was the case previously it will be used here. Furthermore as far as the discussions in this paper are concerned there is no loss in essential generality, compared with previous work (Robinson, 2003), in assuming that the choice of basis of minus 1-forms, (ζ^1, ζ^2) enables the coefficient of ζ^2 in **m** to be set equal to zero.]

Recall that group multiplication in **G** is exterior multiplication and closure is ensured with this type of choice for **m**, that is the vanishing of the coefficient of ζ^2. The identity is the identity in G, 1, and $\mathbf{g}^{-1} = \mathbf{n}^{-1}\mathbf{m}^{-1}$ where $\mathbf{m}^{-1} = 1 - \mu\zeta^1 - \nu\zeta^{12}$.

Under a gauge transformation generated by **g**, $\mathbf{A} \to \hat{\mathbf{A}} = \mathbf{g}^{-1}d\mathbf{g} + \mathbf{g}^{-1}\mathbf{A}\mathbf{g}$, where

$$\begin{aligned}\hat{\mathbf{A}} &= \pi^{-1}d\pi + \pi^{-1}[\alpha + (-k^1\mu)]\pi + \pi^{-1}[\beta_1 + \mu\alpha + \alpha\mu - (k^1)^{-1}\Omega_\mu - k^2\nu]\pi\zeta^1 \\ &\quad + \pi^{-1}[\beta_2 + k^1\nu]\pi\zeta^2 + \pi^{-1}[\gamma + D_\mu\nu + \alpha\nu - \nu\alpha - \mu\beta_2 + \beta_2\mu]\pi\zeta^{12}.\end{aligned} \tag{4}$$

Furthermore $\mathbf{F} \to \hat{\mathbf{F}} = \pi^{-1}\mathbf{m}^{-1}\mathbf{F}\mathbf{m}\pi$ where, if $\mathbf{F} = \overset{2}{F} + \overset{3}{F}_i\zeta^i + \overset{4}{F}\zeta^{12}$,

$$\begin{aligned}\mathbf{m}^{-1}\mathbf{F}\mathbf{m} &= \overset{2}{F} + (\overset{3}{F}_1 + \overset{2}{F}\mu - \mu\overset{2}{F})\zeta^1 + \overset{3}{F}_2\zeta^2 \\ &\quad + (\overset{4}{F} + \overset{2}{F}\nu - \nu\overset{2}{F} + (\overset{3}{F}_2\mu + \mu\overset{3}{F}_2)\zeta^{12}.\end{aligned} \tag{5}$$

It should be noted that with the definition of gauge transformation used here any connection 1-form **A**, as in Eq. (1), is gauge equivalent to the exterior product of a type $N = 1$ form with a minus 1-form. For example, a gauge transformation of $\mathbf{A}$ generated by $\mathbf{m}$, with $\mu = (k^1)^{-1}\alpha$ and $\nu = -(k^1)^{-1}\beta_2$, gives

$$\hat{\mathbf{A}} = [\beta_1 + (k^1)^{-1}k^2\beta_2 + (k^1)^{-1}\Omega_\alpha + (-\gamma + (k^1)^{-1}d_\alpha\beta_2)\zeta^2]\zeta^1. \tag{6}$$

Any flat type $N = 2$ connection can be expressed in the form $\mathbf{g}^{-1}d\mathbf{g}$. A simple computation gives

$$\begin{aligned}\mathbf{g}^{-1}d\mathbf{g} = [\pi^{-1}d\pi + \pi^{-1}(-k^1\mu)\pi] + \pi^{-1}[-(k^1)^{-1}\Omega_\mu - k^2\nu]\pi\zeta^1 \\ + k^1\pi^{-1}\nu\pi\zeta^2 + \pi^{-1}(d_\mu\nu)\pi\zeta^{12},\end{aligned} \tag{7}$$

where d_μ denotes the covariant exterior derivative with respect to $-k^1\mu$, so that $d_\mu\nu = d\nu + (-k^{\mathbf{1}}\mu)\nu - \nu(-k^1\mu)$, and $\Omega_\mu \equiv d(-k^1\mu) + (-k^{\mathbf{1}}\mu)(-k^1\mu)$. It follows from Eq. (7) that the expression for the flat connection given by Eq. (3) is equal to $\mathbf{m}^{-1}d\mathbf{m}$ where

$$\mathbf{m} = \mathbf{1} - (k^1)^{-1}\alpha\zeta^{\mathbf{1}} + (k^1)^{-1}\beta_2\zeta^{12}. \tag{8}$$

The convariant exterior derivative of a generalized p-form $\mathbf{s}$ is defined by using the standard type of formula

$$d_{\mathbf{A}}\mathbf{s} = d\mathbf{s} + \mathbf{A}\mathbf{s} + (-1)^{p+1}\mathbf{s}\mathbf{A},$$

and throughout the paper covariant exterior derivatives are denoted by d with a subscript indicating the connection.

A generalized zero-form $\mathbf{p}$ is said to be parallely-transported along a curve in M, with tangent vector V, when $V \rfloor d_{\mathbf{A}}p = 0$. When $\mathbf{A}$ is flat a generalized zero-form at any point in M defines a unique parallel p-form field on M satisfying the linear system of first-order equations of parallel propagation, $d_{\mathbf{A}}p = 0$.

3. CARTAN'S STRUCTURE EQUATIONS FOR METRIC CONNECTIONS

Recall that if θ is a $n \times 1$ matrix of ordinary 1-forms on M with entries θ^a, $a = 1, \ldots, n$, which constitutes a co-frame, then Cartan's structure equations for an affine connection represented by an $n \times n$ matrix valued 1-form ω are given by

$$\begin{aligned}d_\omega\theta &= \Theta, \\ d\omega + \omega\omega &= \Omega_\omega.\end{aligned} \tag{9}$$

where Θ and Ω are the torsion and curvature 2-forms. The first and second Bianchi identities are given by

$$\begin{aligned}d_\omega\Theta - \Omega_\omega\theta &= 0, \\ d_\omega\Omega_\omega &= 0.\end{aligned} \tag{10}$$

Now consider the case where the co-frame determines a metric of signature (p, q) given by $ds^2 = \eta_{ab}\theta^a \otimes \theta^b$, where η_{ab} are constants. The metric connection 1-forms and curvature 2-forms take values in $so(p, q)$. As is well known, Cartan's equations for metric connections are equivalent to equations expressed entirely in terms of $so(p, q)$-valued forms. This can be seen by introducing the $so(p, q)$-valued 2- and 3-forms, Σ and Ξ, with entries in their respective $n \times n$ matrix representations given by

$$\Sigma^a_b = \theta^a\theta_b,$$
$$\Xi^a_b = \Theta^a\theta_b - \theta^a\Theta_b. \tag{11}$$

The equations

$$d_\omega\Sigma = \Xi,$$
$$d\omega + \omega\omega = \Omega_\omega \tag{12}$$

hold if and only if Cartan's structure equations are satisfied and can be regarded as the $SO(p, q)$ version of those equations. In addition the analogue of the first Bianchi identity is

$$d_\omega\Xi + \Sigma\Omega_\omega - \Omega_\omega\Sigma = 0. \tag{13}$$

Furthermore the connection ω is the unique Levi-Civita torsion free metric connection when Θ vanishes, and $\Theta = 0$ if and only if $\Xi = 0$.

The $SO(p, q)$ Cartan equations can be naturally expressed in terms of type $N = 2$ flat generalized connections. By using the results of Section 2, or by direct calculation, it is a straightforward matter to verify the following results.

Proposition 3.1. *Let the so(p, q)-valued generalized connection 1-form* **A** *be given by*

$$\mathbf{A} = \omega + \Sigma\zeta^1 + [-(k^2)^{-1}\Omega - (k^2)^{-1}k^1\Sigma]\zeta^2 - (k^2)^{-1}\Xi\zeta^{12}, \tag{14}$$

where $\sum^a_b = \theta^a\theta_b$ *and* Ω *and* Ξ *are, respectively, a so(p, q)-valued 2-form and 3-form. Its curvature is given by*

$$\mathbf{F} = \Omega_\omega - \Omega + [d_\omega\Sigma - \Xi]\zeta^1 + (k^2)^{-1}[-d_\omega\Omega - k^1 d_\omega\Sigma + k^1\Xi]\zeta^2$$
$$+ (k^2)^{-1}[-d_\omega\Xi + \Omega\Sigma - \Sigma\Omega]\zeta^{12}. \tag{15}$$

Hence **A** *is flat if and only if* $\Omega = \Omega_\omega$, $\Xi^a_b = d_\omega\theta^a\theta_b - \theta^a d_\omega\theta_b$ *and the Cartan structure equations, Eqs. (11) and (12)—and Eq. (13), are satisfied.*

The next corollary follows directly from Eq. (8).

Corollary 3.1. *When* $\mathbf{A}$ *is flat* $\mathbf{A} = \mathbf{m}^{-1}d\mathbf{m}$ *where*

$$\mathbf{m} = 1 - (k^1)^{-1}\omega\zeta^1 + (k^1k^2)^{-1}[-\Omega_\omega - k^1\Sigma]\zeta^{12}. \tag{16}$$

Corollary 3.2. *When* $\Xi = 0$ *in Eqs. (14) and (15), and the 1-forms* $\{\theta^a\}$ *are linearly independent,* $\mathbf{A}$ *is flat if and only if* ω *is the Levi-Civita connection, with curvature* Ω*, of the signature (p, q) metric* $ds^2 = \eta_{ab}\theta^a\theta^b$.

By imposing restrictions on the choices of Ω and Ξ in Eqs. (14) and (15) different types of geometries can be associated with flat connections. The following corollary contains one example of this—Ricci flat metrics.

Corollary 3.3. *Let (i)* $\Xi = 0$ *in Eqs. (14) and (15), (ii)* $\Omega^a_{\cdot b} = 1/2C^{a\cdot c}_{\cdot b\cdot d}\Sigma^d_{\cdot c}$*, where* $C^{a\cdot c}_{\cdot b\cdot a} = C^{a\cdot b}_{\cdot b\cdot a} = 0$*, and (iii) the 1-forms* $\{\theta^a\}$ *be linearly independent.*

Then the connection 1-form $\mathbf{A}$ is flat if and only if ω is the Levi-Civita connection of the *Ricci-flat* metric $ds^2 = \eta_{ab}\theta^a\theta^b$.

4. FLAT GENERALIZED CONNECTIONS AND FIELD EQUATIONS IN FOUR DIMENSIONS

Here a source-free Yang-Mills field, with internal symmetry group G_I, coupled to gravity through the Einstein–Yang-Mills equations will be considered. For the sake of definiteness four-metrics, $ds^2 = \eta_{ab}\theta^a\theta^b$, with $\eta_{ab} = \mathrm{diag}(1, 1, 1, -1)$ will be considered. The field equations, can be written in the form

$$\begin{aligned} d_\omega\Sigma &= 0,\\ d_\alpha * \mathcal{F} &= 0,\\ d\omega + \omega\omega &= \Omega,\\ d\alpha + \alpha\alpha &= \mathcal{F}, \end{aligned} \tag{17}$$

where ω is the *so*(3,1)-valued Levi-Civita connection 1-form, $\Sigma^a_{\cdot b} = \theta^a\theta_b$, and α is the Yang-Mills gauge potential (connection) and takes values in the Lie algebra of G_I. Here $*\mathcal{F}$ denotes the Hodge dual of the Yang-Mills field (curvature) two form $\mathcal{F} = \frac{1}{2}\mathcal{F}^i{}_{j_{ab}}\Sigma^{ab}$, and here $a, b, c, d = 1 - 4$, and the internal Lie algebra indices $i, j = \mathbf{1} - \mathrm{dim}G_I$. The curvature 2-form Ω is given by

$$\Omega^a_{\cdot b} = \frac{1}{2}C^a_{\cdot bcd}\Sigma^{cd} + 2\pi(T_{bd}\Sigma^{ad} + T_{bc}\Sigma^{ac} + \Sigma^c_{\cdot b}T^a_c + \Sigma^d_{\cdot b}T^a_d), \tag{18}$$

where $C^a_{\cdot bcd}$ are the components of the Weyl conformal curvature of the metric and T_{ab} are the components of the energy–momentum tensor of the Yang-Mills field.

The last equation is just a convenient way of writing Einstein's equations

$$G_{ab} = 8\pi T_{ab},$$
$$T_{ab} = \frac{1}{4\pi} tr\left(F_{ac}F_{b}^{\cdot c} - \frac{1}{4}\eta_{ab}F_{cd}F^{cd}\right). \quad (19)$$

These equations can be rewritten in terms of a connection 1-form Γ and a 2-form Υ which take values in the Lie algebra of $SO(1, 3) \times G_I$ as follows (c.f. Robinson, 1995). Let

$$\Gamma = \omega \mathbf{1}_G + 1_L \alpha,$$
$$\Upsilon = \Sigma \mathbf{1}_G + 1_L * F,$$
$$\Pi = \Omega \mathbf{1}_G + 1_L F, \quad (20)$$

where 1_L and 1_G respectively represent the identities in $SO(\mathbf{1},3)$ and G_I. The first two equations in Eqs. (17) are then equivalent to the equation

$$d_\Gamma \Upsilon = 0, \quad (21)$$

and the second pair of equations in Eq. (17) are equivalent to the equation

$$d\Gamma + \Gamma\Gamma = \Pi. \quad (22)$$

The components of these objects may be written in an index notation as
$\Gamma^{\alpha}_{\cdot\beta} \equiv \Gamma^{a\cdot i}_{\cdot b\cdot j} = \omega^{a}_{\cdot b}\delta^{i}_{j} + \delta^{a}_{\cdot b}\alpha^{i}_{\cdot j}$, $\Upsilon^{\alpha}_{\cdot\beta} \equiv \Upsilon^{a\cdot i}_{\cdot b\cdot j} = \Sigma^{a}_{\cdot b}\delta^{i}_{\cdot j} + \delta^{a}_{\cdot b} * F^{i}_{\cdot j}$,
and $\Pi^{\alpha}_{\cdot\beta} \equiv \Pi^{a\cdot i}_{\cdot b\cdot j} = \Omega^{a}_{\cdot b}\delta^{i}_{\cdot j} + \delta^{a}_{\cdot b}F^{i}_{\cdot j}$.

By comparing the Einstein–Yang-Mills field equations given by Eqs. (21) and (22) with the equations in Section 3, it is clear that these equations can be rewritten in terms of a flat generalized connection 1-form. Let

$$\mathbf{A} = \Gamma + \Upsilon\zeta^1 + [-(k^2)^{-1}\Pi - (k^2)^{-1}k^1\Upsilon]\zeta^2, \quad (23)$$

where $\Gamma \equiv \omega 1_G + \mathbf{1}_L\alpha$, $\Upsilon \equiv \Sigma 1_G + \mathbf{1}_L * (d\alpha + \alpha\alpha)$, and Π are respectively 1-forms and 2-forms with values in the Lie algebra of $SO(1, 3) \times G_I$. Then, since the curvature 2-form $\mathbf{F} = d\mathbf{A} + \mathbf{AA}$ is given by

$$\mathbf{F} = (d\Gamma + \Gamma\Gamma - \Pi) + d_\Gamma\Upsilon\zeta^1 - (k^2)^{-1}[d_\Gamma\Pi + k^1 d_\Gamma\Upsilon]\zeta^2 + (k^2)^{-1}[\Pi\Upsilon - \Upsilon\Pi]\zeta^{12},$$

the following proposition holds

Proposition 4.1. *The connection 1-form* **A** *given by Eq. (23) is flat if and only if the Einstein–Yang-Mills equations are satisfied.*

It is a straightforward matter to show that the Yang-Mills equations alone can also be represented by a flat connection.

Proposition 4.2. *Consider the two parameter family of connection 1-forms*

$$\mathbf{A} = \alpha - D^{-1}(c^2\mathcal{F} + k^2 * \mathcal{F})\zeta^1 + D^{-1}(c^1\mathcal{F} + k^1 * \mathcal{F})\zeta^2, \tag{24}$$

where the parameters c^1, c^2 are chosen so that $D \equiv c^2k^1 - c^1k^2$ is nonzero. Since the curvature 2-forms are given by

$$\mathbf{F} = (d\alpha + \alpha\alpha - \mathcal{F}) - D^{-1}(c^2 d_\alpha\mathcal{F} + k^2 d_\alpha * \mathcal{F})\zeta^1 + D^{-1}(c^1 d_\alpha\mathcal{F} + k^1 d_\alpha * \mathcal{F})\zeta^2,$$

the connection $\mathbf{A}$ is flat if and only if the one form α and the two form $\mathcal{F}$ satisfy the source-free Yang-Mills equations, $d\alpha + \alpha\alpha = \mathcal{F}$ and $d_\alpha * \mathcal{F} = 0$.

5. SOURCE-FREE MAXWELL EQUATIONS FOR TYPE $N = 2$ FORMS

In this concluding section it will be shown that the structure of the source-free Maxwell-like equations for type $N = 2$ forms differs from the structure of the standard Maxwell equations. This is in line with previous results for $N = 1$ forms (Nurowski and Robinson, 2002). It illustrates a difference between the fields and equations considered above and those determined by nonflat generalized connections.

The generalized source-free Maxwell equations for type $N = 2$ forms, in a four-dimensional space–time with a Lorentzian metric, are defined to be

$$d\mathbf{F} = 0, \tag{25}$$

$$d \star \mathbf{F} = 0, \tag{26}$$

where $\mathbf{F}$ is a generalized 2-form and $\star\mathbf{F}$ is its Hodge dual, a generalized zero-form. For $N > 0$ all closed generalized forms are exact, hence Eq. (25) can be solved by introducing a potential for $\mathbf{F}$, a generalized 1-form $\mathbf{A}$ with $\mathbf{F} = d\mathbf{A}$. As usual $\mathbf{A}$ can be interpreted as a connection 1-form with curvature $\mathbf{F}$ and the gauge freedom $\mathbf{A} \to \hat{\mathbf{A}} = \mathbf{g}^{-1}d\mathbf{g} + \mathbf{A}$ is the one-dimensional (abelian) version of the gauge transformation given in Section 2. Using the notation of Section 2, applied to this particular potential, it follows from Eqs. (1) and (2) that if $\mathbf{A} = \alpha + \beta_i\zeta^i + \gamma\zeta^{12}$ then

$$\mathbf{F} = (d\alpha + \beta_i k^i) + (d\beta_1 + k^2\gamma)\zeta^1 + (d\beta_2 - k^1\gamma)\zeta^2 + d\gamma\zeta^{12}. \tag{27}$$

Using the definition of Hodge dual for generalized 2-forms (Robinson, 2003).

$$\star\mathbf{F} = *d\gamma + (*d\beta_2 - k^1 * \gamma)\zeta^1 - (*d\beta_1 + k^2 * \gamma)\zeta^2 + (*d\alpha + k^i * \beta_i)\zeta^{12}. \tag{28}$$

Now Eq. (26) is satisfied if and only if

$$d \star \mathbf{F} = \overset{1}{\chi} + \overset{2}{\chi}_i\zeta^i + \overset{3}{\chi}\zeta^{12}, \tag{29}$$

is zero, that is if and only if

$$\overset{1}{\chi} \equiv d(*d\gamma) + [(k^1)^2 + (k^2)^2] * \gamma - k^1 * d\beta_2 + k^2 * d\beta_1 = 0; \tag{30}$$

$$\overset{2}{\chi}_1 \equiv d(*d\beta_2 - k^1 * \gamma) - k^2(*d\alpha + k^i * \beta_i) = 0; \tag{31}$$

$$\overset{2}{\chi}_2 \equiv -\{d(*d\beta_1 + k^2 * \gamma) - k^1(*d\alpha + k^i * \beta_i)\} = 0; \tag{32}$$

$$\overset{3}{\chi} \equiv d(*d\alpha + k^i * \beta_i) = 0. \tag{33}$$

It should be noted that Eq. (33) is satisfied whenever either Eq. (31) or Eq. (32) is satisfied. By using the gauge freedom discussed in Section 2 it is possible to choose a gauge in which $\alpha = \beta_2 = 0$. The remaining gauge freedom then is given by gauge transformations in which, $\mathbf{g} = \mathbf{mn}$, $\mathbf{n} = \pi$, $\mathbf{m} = 1 + (k^1)^{-1} d(\ln \pi)\zeta^1$, and under which β_1 and γ remain unchanged. In this gauge it follows from Eq. (31) that

$$\beta_1 = (k_2)^{-1}\delta\gamma, \tag{34}$$

where $\delta = *d*$ is the co-differential operator. When this expression for β_1 is used Eq. (32) can be seen to satisfied whenever Eq. (30) is; and the latter, now the only equation remaining to be solved, becomes the Proca-type equation

$$\Delta\gamma + [(k^1)^2 + (k^2)^2]\gamma = 0, \tag{35}$$

where $\Delta = d\delta + \delta d$ is the Laplace operator for 3-forms. Consequently, in this gauge the potential is given by $\mathbf{A} = (k_2)^{-1}\delta\gamma\zeta^1 + \gamma\zeta^{12}$, and it satisfies the field equations, Eqs. (25) and (26), if and only if Eq. (35) is satisfied.

The above calculations could be extended, for example, to straightforwardly generalize the formalism of higher-order gauge theories (Alvarez and Olive, 2003) in *n* dimensions.

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